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ON SOME SEQUENCES DEFINED BY RECURRENCE RELATIONS
OF INCREASING LENGTH

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On some sequences defined by recurrence relations of increasing length

by

J. van de Lune

ABSTRACT

This note contains a discussion of some convexity properties of the sequence $\{s_n\}_{n=0}^{\infty}$ defined by $s_0 := 1$ and the recurrence relation $s_n = s_0 n^{-\alpha} + s_1(n-1)^{-\alpha} + \ldots + s_{n-2}2^{-\alpha} + s_{n-1}$, $n > 0$, α being any positive constant.

KEY WORDS & PHRASES: *Recurrence relations, convexity, zeros, special functions*

0. INTRODUCTION

Recently, the error analysis of a numerical procedure for the solution of a certain type of integral equations led us to the problem of determining the asymptotic behaviour of some sequences defined by linear recurrence relations of increasing length.

More precisely, let $\{g_n\}_{n=0}^{\infty}$ be a given non-negative sequence such that $\Sigma_{n=0}^{\infty} g_n > 1$ (divergence being permitted) whereas the corresponding power series $\Sigma_{n=0}^{\infty} g_n z^n$, $z \in \mathbb{C}$, has radius of convergence 1 (say). Let $s_0 := 1$ and define recursively for $n > 0$

$$(1) \qquad s_n := s_0 g_{n-1} + s_1 g_{n-2} + \ldots + s_{n-2} g_1 + s_{n-1} g_0.$$

In particular, we had $g_n = (n+1)^{-\alpha}$ for $n \geq 0$, α being any (fixed) positive number, and we wanted to know the asymptotic behaviour of $\{s_n\}_{n=0}^{\infty}$ as $n \to \infty$. In order to attack this problem we define

$$G(z) := \sum_{n=0}^{\infty} g_n z^n, \qquad z \in \mathbb{C},\ |z| < 1,$$

and observe that $x\,G(x)$ increases from 0 to $\Sigma_{n=0}^{\infty} g_n > 1$ as x increases from 0 to 1, so that there exists a unique $x_0 \in (0,1)$ such that $x_0\,G(x_0) = 1$. Now define $a_n := s_n x_0^n$ and $p_n := g_n x_0^{n+1}$ for $n \geq 0$, so that $a_0 = 1$, $p_n \geq 0$ for $n \geq 0$ and $\Sigma_{n=0}^{\infty} p_n = 1$. From this it is clear that we may define $P(z) := \Sigma_{n=0}^{\infty} p_n z^n$, $z \in \mathbb{C}$, $|z| < 1/x_0$, and that the recurrence relation (1) is equivalent to

$$(2) \qquad a_n := a_0 p_{n-1} + a_1 p_{n-2} + \ldots + a_{n-2} p_1 + a_{n-1} p_0 \quad \text{for } n > 0.$$

By mathematical induction it is easily shown that $0 < a_n \leq 1$ for all $n \geq 0$ so that we may define

$$A(z) := \sum_{n=0}^{\infty} a_n z^n, \qquad z \in \mathbb{C},\ |z| < 1.$$

Since $p_n \geq 0$ and $\Sigma_{n=0}^{\infty} p_n = 1$ it follows (by the maximum modulus theorem) that $|zP(z)| < 1$ for $|z| < 1$, so that $1 - zP(z)$ has no zeros in the disc $|z| < 1$.

By means of (2) it is easily shown that

$$A(z)P(z) = \frac{A(z)-1}{z}, \qquad 0 < |z| < 1,$$

so that

$$A(z) = \frac{1}{1-zP(z)}, \qquad |z| < 1,$$

or, equivalently,

$$S(z) := \sum_{n=0}^{\infty} s_n z^n = \frac{1}{1-zG(z)}, \qquad |z| < x_0.$$

Note that the power series for $G(z)$ around $z = 0$ has radius of convergence 1 and that $p_n \geq 0$ for $n \geq 0$, so that, by a well-known theorem of Pringsheim (cf. TITCHMARSH [4; pp. 214-215]), the point $z = 1$ is a singular point of $G(z)$ and hence of $S(z)$. A similar remark holds true for $A(z)$ with respect to the point $z = 1/x_0$. In order to study the sequences $\{s_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ in greater detail we will make use of Cauchy's formula

$$s_n = \frac{1}{2\pi i} \oint \frac{S(z)}{z^{n+1}} dz = \frac{1}{2\pi i} \oint \frac{1}{1-zG(z)} \frac{dz}{z^{n+1}},$$

where $\oint$ denotes counter clockwise integration along a circle around $z = 0$ with positive radius $\rho < x_0$.

In the following sections of this note we will be exclusively concerned with the specific case $g_n = (n+1)^{-\alpha}$, $\alpha > 0$, mentioned above. Besides determining the "main term" of the asymptotic behaviour of s_n, our main goal will be to show that a_n *tends logarithmically convex to a limit* L (to be specified later on) and we already note here that the analytic continuation of $P(z)$ will play an intriguing role in our discussion.

1. THE CASE $\alpha = 1$

As a *model* for our considerations we first consider the case in which $g_n = \frac{1}{n+1}$ for $n \geq 0$. It is clear that

$$G(z) = \sum_{n=0}^{\infty} \frac{z^n}{n+1} = -\frac{1}{z}\log(1-z), \qquad |z| < 1,$$

from which it is easily seen that

$$x_0 = 1 - e^{-1} (\cong .632\ 120\ 559).$$

The generating function of $\{s_n\}_{n=0}^{\infty}$ is in this case

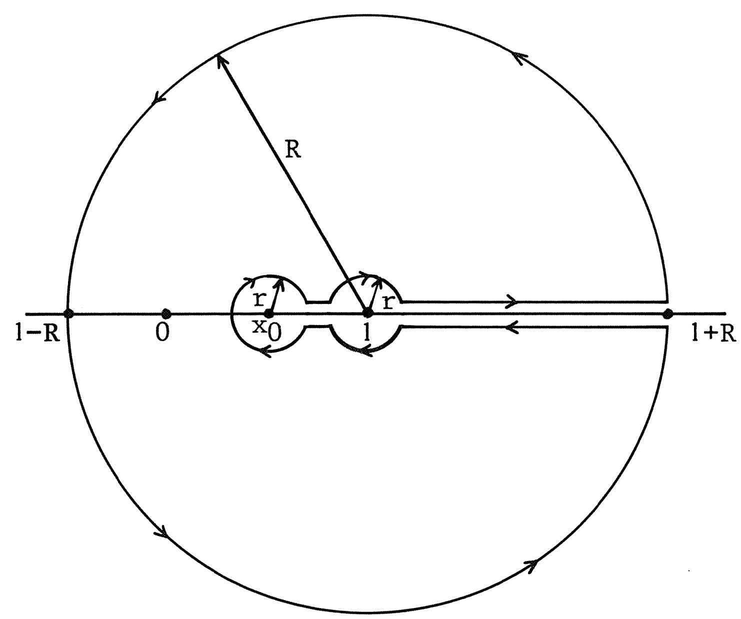
$$S(z) := \sum_{n=0}^{\infty} s_n z^n = \frac{1}{1-zG(z)} = \frac{1}{1+\log(1-z)},$$

so that Cauchy's formula for s_n reads

$$s_n = \frac{1}{2\pi i} \oint \frac{1}{1+\log(1-z)} \frac{dz}{z^{n+1}}, \tag{3}$$

where $\oint$ denotes counter clockwise integration along a circle around $z = 0$ with positive radius $\rho < x_0$.

Now choose a small positive r and a large positive R and deform the contour of integration in (3) as depicted below:



This deformation of the contour in (3) is possible due to the fact that $1 + \log(1-z)$ has $z = x_0$ as its only (simple) zero and $z = 1$ as its only singularity (on the entire Riemann surface corresponding to $1 + \log(1-z)$). Since $|\log(1-z)|$ tends uniformly to infinity for $z \to 1$ as well as for $|z| \to \infty$ *(so that* $\frac{1}{1+\log(1-z)}$ *is bounded for* $z \to 1$ *as well as for* $|z| \to \infty$), we have by a standard argument

$$s_n = \frac{\Lambda}{x_0^{n+1}} + \int_0^\infty \left(\frac{1}{1+\log u-\pi i} - \frac{1}{1+\log u+\pi i}\right)\frac{du}{(1+u)^{n+1}}$$

so that

$$(4) \qquad a_n = \frac{\Lambda}{x_0} + x_0^n \int_0^\infty \frac{1}{(1+\log u)^2+\pi^2}\,\frac{du}{(1+u)^{n+1}},$$

where Λ is such that $-\Lambda$ is the residue of $S(z) = \frac{1}{1+\log(1-z)}$ at $z = x_0$. It is easily verified that

$$(5) \qquad \Lambda = 1 - x_0 (\cong .367\ 879\ 441).$$

By the general theory of log-convex functions (cf. ARTIN [1]) it is immediately clear from (4) and (5) that $\{a_n\}_{n=0}^\infty$ tends log-convex to its limit

$$L := \frac{\Lambda}{x_0} = \frac{1-x_0}{x_0} = \frac{1}{e-1} \ (\cong .581\ 976\ 707).$$

2. THE CASE $\alpha \in (0,1)$

For any (fixed) $\alpha \in (0,1)$ we now take $g_n = (n+1)^{-\alpha}$ for $n \geq 0$. Since

$$G(z) = 1 + \frac{z}{2^\alpha} + \frac{z^2}{3^\alpha} + \ldots, \qquad |z| < 1,$$

the number x_0 is determined by

$$1 = x_0 + \frac{x_0^2}{2^\alpha} + \frac{x_0^3}{3^\alpha} + \ldots .$$

(Note that $x_0 = x_0(\alpha)$ increases from $\frac{1}{2}$ to 1 as α increases from 0 to ∞.)

In order to proceed here similarly as in Section 1 we first derive an expression for $G(z)$ which throws some light on the analytic continuation of $G(z)$. For $\alpha > 0$ we have

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt,$$

so that (by the substitution $t = nu$, $n > 0$)

$$\frac{1}{n^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-nu} u^{\alpha-1} du,$$

and summation over n leads to

$$G(z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{u^{\alpha-1}}{e^u - z} du. \tag{6}$$

From this representation it is clear that $G(z)$ can be continued analytically to the slit plane $\mathbb{C}^* := \mathbb{C}\backslash[1,\infty)$. Our first application of (6) is to show that x_0 is the only zero of $1 - zG(z)$ in $\mathbb{C}^*$. In order to see this we write $z = x + yi$ and consider imaginary parts in the equation $zG(z) = 1$.

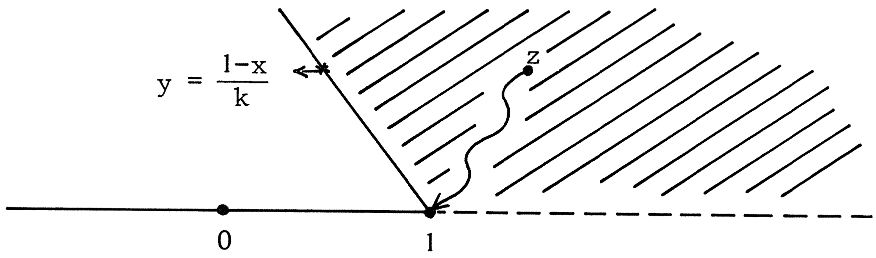
Since

$$\operatorname{Im}(zG(z)) = \frac{y}{\Gamma(\alpha)} \int_0^\infty \frac{e^u u^{\alpha-1}}{(e^u-x)^2+y^2} du,$$

it follows that $zG(z) = 1$ is impossible for $y \neq 0$. However, on the real axis, for $x < 1$, the function $xG(x)$ is clearly increasing so that x_0 is the only zero of $1 - zG(z)$ in the domain $\mathbb{C}^*$. Later on we will also show that the extension of $1 - zG(z)$ to the edges of the slit $(1,\infty)$ does not vanish there.

In Section 1 we used that $|\log(1-z)| \to \infty$ (uniformly) for $z \to 1$ as well as for $|z| \to \infty$. With this in mind we now show that $|zG(z)| \to \infty$ (uniformly) as $z \to 1$ $(z \in \mathbb{C}^*)$.

It is clear that, where convenient, we may just as well show that $|G(z)| \to \infty$ (uniformly) as $z \to 1$. Again, write $z = x + yi$ and let z tend to 1 with z belonging to the sector: $y > 0$, $y \geq \frac{1-x}{k}$ where k is any positive constant. See figure below.



Considering the imaginary part of zG(z) we have

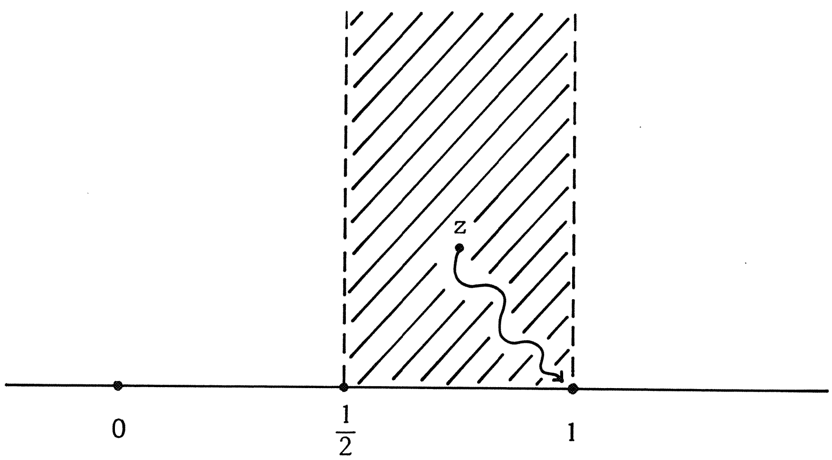
$$\mathrm{Im}(zG(z)) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{ye^u}{(e^u-x)^2+y^2}\, u^{\alpha-1} du = (\text{substitute } u = \log(x+yt))$$

$$= \frac{1}{\Gamma(\alpha)} \int_{\frac{1-x}{y}}^{\infty} \frac{\log^{\alpha-1}(x+yt)}{t^2+1}\, dt,$$

and, observing that the integrand is positive on the whole interval of integration and that $\frac{1-x}{y} \leq k$, it follows that

$$\Gamma(\alpha).\mathrm{Im}(zG(z)) > \int_k^{2k} \frac{\log^{\alpha-1}(x+yt)}{t^2+1}\, dt > \frac{k}{4k^2+1} \log^{\alpha-1}(x+2ky),$$

which (due to $x + 2ky > 1$ and $\alpha < 1$) tends to infinity as $x \to 1$ and $y \to 0$. Now let $z = x + yi$ be restricted to the strip $\frac{1}{2} < x < 1$, $y > 0$ and consider $\mathrm{Re}(G(z))$.



By the substitution $u = \log(xt)$ in

$$\Gamma(\alpha).\mathrm{Re}(G(z)) = \int_0^\infty \frac{e^u-x}{(e^u-x)^2+y^2} u^{\alpha-1} du,$$

we obtain (note that we let $x \to 1$)

$$\Gamma(\alpha).\mathrm{Re}(G(z)) = \int_{\frac{1}{x}}^{\infty} \frac{xt-x}{(xt-x)^2+y^2} (\log xt)^{\alpha-1} \frac{dt}{t} > \int_{\frac{1}{x}}^{2} =$$

$$= \frac{1}{x} \int_{\frac{1}{x}}^{2} \frac{t-1}{(t-1)^2+(\frac{y}{x})^2} (\log xt)^{\alpha-1} \frac{dt}{t}$$

$$> \frac{1}{2} \log^{\alpha-1}(2x) \int_{\frac{1}{x}}^{2} \frac{t-1}{(t-1)^2+(\frac{y}{x})^2} dt$$

$$> c \cdot \int_{\frac{1-x}{y}}^{\frac{x}{y}} \frac{u}{u^2+1} du$$

$$= \frac{c}{2}(\log(x^2+y^2) - \log(y^2+(1-x)^2)), \quad (c = \frac{1}{2}\log^{\alpha-1} 2),$$

from which it is clear that $\mathrm{Re}(G(z)) \to \infty$ as $x \to 1$ and $y \to 0$.

The case $y = 0$, $x \uparrow 1$ can be dealt with directly from the original power series for $G(z)$. Since $G(\bar{z}) = \overline{G(z)}$, this proves that $|zG(z)| \to \infty$ (uniformly) for $z \to 1$. For another proof we refer to MAGNUS, OBERHETTINGER & SONI [3; pp. 32-35].

Our next goal is to show that $|zG(z)| \to \infty$ (uniformly) as $|z| \to \infty$. Although we realize that a substantial part of this rather technical problem has been considered before by FORD [2; Chapter III, Theorem IV, pp. 26-27], we choose to achieve our goal independently. First we restrict $z = x + yi$ to the half plane $x \le 0$ and we will show that $-\mathrm{Re}(zG(z)) \to \infty$ (uniformly) as $|z| \to \infty$.

Writing $p := -x$ and $r := |z|$ we have

$$-\mathrm{Re}(zG(z)) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{-x(e^u-x)+y^2}{(e^u-x)^2+y^2} u^{\alpha-1} du$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{pe^u+r^2}{(e^u+p)^2+y^2} u^{\alpha-1} du$$

$$> \frac{1}{\Gamma(\alpha)} \int_0^{\log r} \frac{r^2}{(r+r)^2+r^2} u^{\alpha-1} du = \frac{\log^\alpha r}{5\Gamma(\alpha+1)},$$

which tends to infinity as $r \to \infty$.

Now restrict $z = x + yi$ to the first quadrant such that $x > 0$ and $y \geq kx$, where k is some positive constant. We will show that also in this case $-\mathrm{Re}(zG(z)) \to \infty$ (uniformly) as $|z| \to \infty$.

As before we have

$$\Gamma(\alpha).\mathrm{Re}(zG(z)) = \int_0^\infty \frac{x(e^u-x)-y^2}{(e^u-x)^2+y^2} u^{\alpha-1} du.$$

We first show that the tail $\int_{\log(r^2/x)}^\infty$ of this integral is bounded for $r \to \infty$. Substituting $t = xe^u/r^2$ and observing that the integrand is positive on $(\log(r^2/x),\infty)$ we obtain

$$0 < \int_{\log \frac{r^2}{x}}^\infty < \int_{\log \frac{r^2}{x}}^\infty \frac{xe^u}{(e^u-x)^2+y^2} u^{\alpha-1} du$$

$$= \int_1^\infty \frac{r^2 t}{(\frac{r^2}{x}t-x)^2+y^2} \log^{\alpha-1}(\frac{r^2}{x}t)\frac{dt}{t} < \qquad \text{(note that } 0 < \alpha < 1)$$

$$< \log^{\alpha-1}(\frac{r^2}{x}) \int_1^\infty \frac{r^2}{(\frac{r^2}{x}t-x)^2+y^2} dt = (\frac{r^2}{x}t - x = yu)$$

$$= \log^{\alpha-1}(\frac{r^2}{x}) \int_{\frac{\frac{r^2}{x}-x}{y}}^\infty \frac{r^2}{y^2(u^2+1)} \frac{ydu}{r^2/u} = \frac{x}{y} \log^{\alpha-1}(\frac{r^2}{x}) \int_{\frac{\frac{r^2}{x}-x}{y}}^\infty \frac{du}{u^2+1}$$

$$< \frac{x}{y} \log^{\alpha-1}(\frac{r^2}{x}) \int_{-\infty}^\infty \frac{du}{u^2+1} .$$

Since $\frac{x}{y} \le \frac{1}{k}$ and $\log^{\alpha-1}(\frac{r^2}{x}) \to 0$ as $r \to \infty$ it follows that "the tail" is bounded as $r \to \infty$. Hence, in case $x > 0$ and $y \ge kx$ we may complete the proof of our claim as follows.

$$\left| \int_0^{\log\frac{r^2}{x}} \right| = \int_0^{\log\frac{r^2}{x}} \frac{r^2 - xe^u}{(e^u - x)^2 + y^2}\, u^{\alpha-1} du > \int_0^{\log\frac{r^2}{x}} \frac{r^2 - xe^u}{e^{2u} + r^2}\, u^{\alpha-1} du$$

$$= (e^u = rt) = \int_{\frac{1}{r}}^{\frac{r}{x}} \frac{r^2 - xrt}{r^2(t^2+1)} \log^{\alpha-1}(rt)\frac{dt}{t} > \int_{\frac{1}{r}}^{1} >$$

$$> \log^{\alpha-1} r \int_{\frac{1}{r}}^{1} \frac{1-t}{t(t^2+1)}\, dt > \log^{\alpha-1} r \int_{\frac{1}{r}}^{\frac{1}{2}} > \frac{1}{2}\log^{\alpha} r$$

if r is large enough.

It remains to show that $|zG(z)| \to \infty$ (uniformly) as $|z| \to \infty$ with $z = x + yi$ restricted to the sector $x > 0$, $0 < y \le kx$ for some positive k. A real approach, as performed above, appears to be quite cumbersome so that we proceed by complex analytical means in this case. We first aboserve that the representation

$$\Gamma(\alpha).zG(z) = \int_0^\infty \frac{z}{e^u - z}\, u^{\alpha-1} du$$

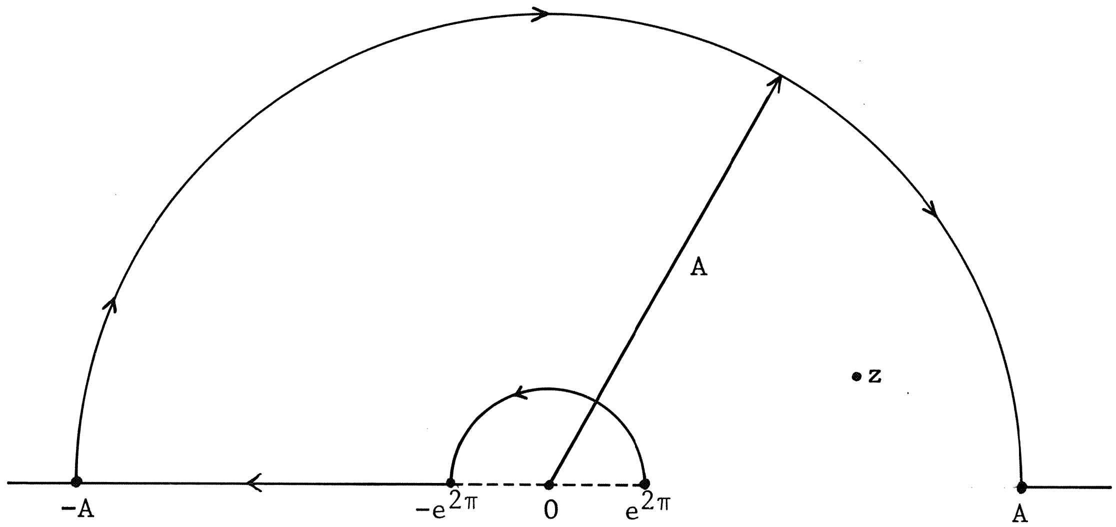
may be transformed (by $e^u = t$) into

$$\Gamma(\alpha).zG(z) = \int_1^\infty \frac{z}{t(t-z)} \log^{\alpha-1} t\, dt = \left(\int_1^{e^{2\pi}} + \int_{e^{2\pi}}^\infty \right),$$

where $\int_1^{e^{2\pi}} \frac{z}{t(t-z)} \log^{\alpha-1} t\, dt$ is easily seen to be bounded for $|z| \to \infty$. It remains to show that $|\int_{e^{2\pi}}^\infty| \to \infty$ (uniformly) as $|z| \to \infty$. Noting that

$$\int_{e^{2\pi}}^{\infty} = \lim_{A\to\infty} \int_{e^{2\pi}}^{A}$$

and replacing the integration from $e^{2\pi}$ to A (> r) along the real axis by the contour depicted below



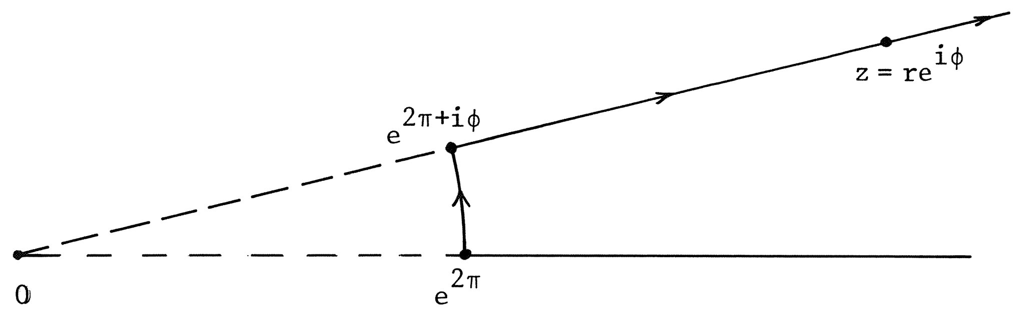
we easily obtain (by letting $A \to \infty$)

$$\int_{e^{2\pi}}^{\infty} = \oint_{e^{2\pi}}^{-e^{2\pi}} + \int_{-e^{2\pi}}^{-\infty} + 2\pi i \log^{\alpha-1} z.$$

Since $\oint_{e^{2\pi}}^{-e^{2\pi}}$ and $\log^{\alpha-1} z$ are bounded for $|z| \to \infty$ it remains to show that the integral

$$- \int_{-e^{2\pi}}^{-\infty} = \int_{e^{2\pi}}^{\infty} \frac{z}{u(u+z)} \log^{\alpha-1}(-u)\,du$$

tends uniformly to infinity as $|z| \to \infty$. Replace the contour from $e^{2\pi}$ to ∞ along the real axis by the contour depicted below.



Then the integral along the circular curve from $e^{2\pi}$ to $e^{2\pi+i\phi}$ is easily seen to be bounded as $|z| \to \infty$, whereas for the remaining integral from $e^{2\pi+i\phi}$ to $\infty . e^{i\phi}$ we have

$$\int_{e^{2\pi}}^{\infty} \frac{re^{i\phi}}{we^{i\phi}(we^{i\phi}+re^{i\phi})} \log^{\alpha-1}(-we^{i\phi})e^{i\phi}dw$$

$$= \int_{e^{2\pi}}^{\infty} \frac{r}{w(w+r)} \log^{\alpha-1}(-we^{i\phi})dw.$$

For the real part of this integral we have

$$\int_{e^{2\pi}}^{\infty} \frac{r}{w(w+r)}(\log^2 w + (\pi+\phi)^2)^{\frac{\alpha-1}{2}} \cos((1-\alpha)\frac{\phi+\pi}{\log w})dw$$

$$> (\cos((1-\alpha)\frac{\phi+\pi}{2\pi})) \int_{e^{2\pi}}^{\infty} \frac{r}{w(w+r)}(2\log^2 w)^{\frac{\alpha-1}{2}} dw$$

$$> 2^{\frac{\alpha-1}{2}} \cos(1-\alpha) \int_{e^{2\pi}}^{r} \frac{r}{w(r+r)} \log^{\alpha-1} r \, dw >> \log^{\alpha} r,$$

which tends to infinity as $r \to \infty$.

In order to complete our line of argument of Section 1 we finally prove that the (continuous) extension of the function $zG(z)$ does not assume the value 1 on the edges of the slit $(1,\infty)$. In order to see this we observe that for $x > 1$ and $y > 0$,

$$\Gamma(\alpha).\mathrm{Im}(zG(z)) = \int_0^\infty \frac{ye^u}{(e^u-x)^2+y^2}\, u^{\alpha-1}du = \int_{\frac{1-x}{y}}^\infty \frac{\log^{\alpha-1}(x+yt)}{t^2+1}\, dt$$

$$> \int_0^\infty \frac{\log^{\alpha-1}(x+yt)}{t^2+1}\, dt,$$

so that (by Lebesgue's dominated convergence theorem)

$$\liminf_{y \downarrow 0} \mathrm{Im}(zG(z)) \geq \frac{\log^{\alpha-1} x}{\Gamma(\alpha)} \frac{\pi}{2} > 0.$$

Similarly as in Section 1 we may conclude that for any $\alpha \in (0,1)$ there exists a positive constant $L = L(\alpha)$ and a positive function $f(u) = f_\alpha(u)$ for $u > 1$ such that

$$a_n = a_n(\alpha) = L + x_0^n \int_1^\infty f(u) \frac{du}{u^{n+1}},$$

and the log-convexity of $\{a_n\}_{n=0}^\infty$ follows as before.

As to the value of L one may verify that

$$L = \frac{\Lambda}{x_0} = \frac{1}{x_0} \cdot \frac{1}{G(x_0)+x_0G'(x_0)} = \frac{1}{1+x_0^2G'(x_0)},$$

where $-\Lambda$ is the residue of $\frac{1}{1-zG(z)}$ at the point $z = x_0$.

3. THE CASE $\alpha > 1$

Most of the analysis of Section 2 may easily be adapted to the case $\alpha > 1$. However, for $z \to 1$ we now have

$$\lim_{z\to 1} zG(z) = \zeta(\alpha), \tag{7}$$

and since $\zeta(\alpha) > 1$ the function $\frac{1}{1-zG(z)}$ is bounded for $z \to 1$, so that the method of Section 1 applies here as well. In order to prove (7) we write

$$F(z) := z\,\mathcal{G}(z) - \zeta(\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{z}{e^u - z} - \frac{1}{e^u - 1}\right) u^{\alpha-1} du$$

$$= \frac{z-1}{\Gamma(\alpha)} \int_0^\infty \frac{e^u}{(e^u - z)(e^u - 1)} u^{\alpha-1} du = (e^u = t)$$

$$= \frac{z-1}{\Gamma(\alpha)} \int_1^\infty \frac{\log^{\alpha-1} t}{(t-z)(t-1)} dt \quad (z = 1 + w,\ t = u + 1)$$

$$= \frac{w}{\Gamma(\alpha)} \int_0^\infty \frac{\log^{\alpha-1}(1+u)}{(u-w)u} du.$$

Since $\alpha - 1 > 0$ it is easily seen that

$$F(z) = \frac{2\pi i}{\Gamma(\alpha)} \log^{\alpha-1}(1+w) + \frac{w}{\Gamma(\alpha)} \int_0^{-\infty \cdot w} \frac{\log^{\alpha-1}(1+u)}{(u-w)u} du,$$

so that it remains to show that w times the last integral (I) tends to zero as $w \to 0$. By the transformation $u = -tre^{i\phi}$ $(w = re^{i\phi})$ we have

$$-wI = -w \int_0^{-\infty w} = -re^{i\phi} \int_0^\infty \frac{\log^{\alpha-1}(1-tre^{i\phi})}{(-tre^{i\phi} - re^{i\phi})(-tre^{i\phi})} (-re^{i\phi}) dt$$

$$= \int_0^\infty \frac{\log^{\alpha-1}(1-tre^{i\phi})}{(t+1)t} dt = \int_0^1 + \int_1^\infty .$$

It is easy to show that $\int_1^\infty \to 0$ as $r \to 0$. For the remaining integral $\int_0^1$ we have

$$\left|\int_0^1\right| \ll \int_0^1 \frac{(rt)^{\alpha-1}}{(t+1)t} dt < \int_0^1 \frac{(rt)^{\alpha-1}}{t} dt = r^{\alpha-1} \int_0^1 t^{\alpha-2} dt = \frac{r^{\alpha-1}}{\alpha-1} ,$$

which tends to zero as $r \to 0$. Hence, also in case $\alpha > 1$, the sequence a_n tends log-convex to its limit, the integral expression for a_n being formally the same as in Section 2.

4. THE CASE α COMPLEX

In this section we devote a few words to the case in which α is complex with $\mathrm{Re}(\alpha) > 0$. As before, the function

$$zG(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}, \qquad |z| < 1, \ \mathrm{Re}(\alpha) > 0,$$

has an analytic extension to $\mathbb{C}^* = \mathbb{C}\backslash[1,\infty)$ by means of the formula

$$zG(z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{z}{e^u - z} u^{\alpha-1} du.$$

It is to be expected that, in general the sequence $\{a_n\}_{n=0}^\infty$ will be complex for complex α so that there is no obvious generalization of the log-convexity properties discussed in the previous sections. Moreover, the "main terms" in the description of the asymptotic behaviour of s_n and a_n become more complicated in case α is complex, due to the (numerically observed) fact that the equation

$$zG(z) = 1$$

can have more than one zero in $\mathbb{C}^*$. For example, writing $z = x + yi$, the above equation has *at least* two different solutions for $\alpha = \frac{1}{2} + 7i$:

$$x_1 \cong .815\ 881, \qquad y_1 \cong -.266\ 304$$

and

$$x_2 \cong -.590\ 967, \qquad y_2 \cong .781\ 403.$$

Note that both solutions even lie in the unit disc $|z| < 1$ ($|z_1| \cong .858\ 242$ and $|z_2| \cong .979\ 710$). We have not pursued this topic any further.

5. SOME RELATED PROBLEMS

PROBLEM I. Let $p\colon \mathbb{R}^+ \to \mathbb{R}^+$ be (Lebesgue) measurable such that $p \notin L_1(\mathbb{R}^+)$ whereas

$$\int_0^\infty p(u)e^{-u}du < \infty.$$

Defining

$$\phi(z) = \int_0^\infty \frac{z}{e^u - z}\, p(u)du, \qquad z \in \mathbb{C}^*,$$

we wonder what can be said (under suitable conditions) about the asymptotic behaviour of $\phi(z)$ as $z \to 1$, $z \in \mathbb{C}^*$, or as $|z| \to \infty$, $z \in \mathbb{C}^*$.

PROBLEM II. Does the function

$$\phi(z) := \phi_\alpha(z) := 1 - \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{z}{e^u - z}\, u^{\alpha-1}du, \qquad z \in \mathbb{C}^*,$$

always (i.e. for any fixed $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$) have at least one zero in the disc $|z| < 1$? Are *all* zeros of $\phi(z)$ simple?

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